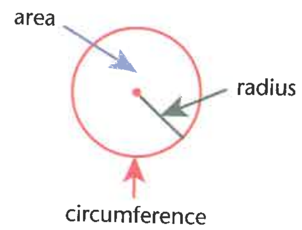
Unit 4: Solve literal equations by changing the subject of the formula



A circle

Mathematical and scientific formulae are examples of literal equations.

Area of circle:
$A = \pi r^2$ (where A is the subject of the formula).

Changing the subject of the area formula gives us:
$r = \sqrt{\frac{A}{\pi}}$ (where r is now the subject of the formula).

Circumference of circle:
$C = 2\pi r$ (where C is the subject of the formula).

Changing the subject of the circumference formula gives ús:
$r = \frac{C}{2\pi}$ (where r is now the subject of the formula).

Literal equations have one (or more) unknown constants, such as r. The solutions are not numerical because of the unknown constant(s). Below are three variations of the same formula:

- $A = l \times b$ – used to determine the area, if the length and breadth are known.
- $b = \frac{A}{l}$ – used to determine the breadth, if the area and length are known.
- $l = \frac{A}{b}$ – used to determine the length, if the area and breadth are known.

If the area is given by $A = 30\text{ m}^2$ and the length by $l = 10$ m, then:

$$\begin{aligned} b &= \frac{A}{l} \\ &= \frac{30}{10} \\ &= 3\text{ m} \end{aligned}$$

Now consider the simple interest formula $A = P(1 + in)$ that is used in financial mathematics:

- A – the amount of money the investment will be worth in n years' time.
- P – the amount of money at the start of the investment.
- i – the interest rate as a decimal (that is, if the interest is 12,3%, then i is 0,123).
- n – the number of years for which the money is invested.

If you are given the values of P, i and n, you can change the subject of the formula to find the value of A. In order to find P, i or n, you must change the subject of the formula as needed.

To find P, change the subject of the formula to get:
$A = P(1 + in)$, thus $P = \frac{A}{1 + in}$.

To find i, change the subject of the formula to get:
$A = P(1 + in) = P + Pin$, thus $Pin = A - P$ and $i = \frac{A - P}{Pn}$.

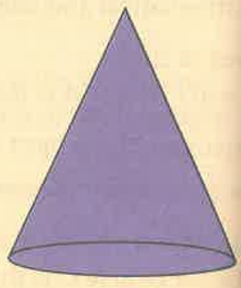
To find n, change the subject of the formula to get:
$A = P + Pin$, thus $Pin = A - P$ and $n = \frac{A - P}{Pi}$.

KEY CONCEPT

'Solve for r' means that you need to *make r the subject of the formula.*

WORKED EXAMPLES

1. Solve for r, if $V = \frac{1}{3}\pi r^2 h$ (the formula to determine the volume of a cone).
   h – the height of the cone.
   r – the radius of the circular base.
   V – the volume of the cone.



A cone

2. Make l the subject of the formula, if $T = 2\pi\sqrt{\frac{l}{g}}$.
   T – the time taken for a pendulum to complete one swing.
   l – the length of the pendulum.
   g – acceleration due to gravity.

SOLUTIONS

1. $V = \frac{1}{3}\pi r^2 h$

   $3V = \pi r^2 h$ → Multiply both sides by 3.

   $\frac{3V}{\pi h} = \frac{\pi r^2 h}{\pi h}$ → Divide both sides by πh.

   $r^2 = \frac{3V}{\pi h}$

   $r = \sqrt{\frac{3V}{\pi h}}$ → Square root both sides.

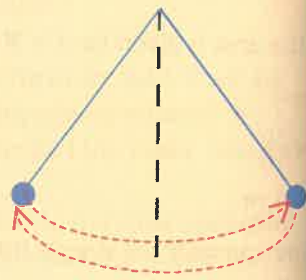
2. $T = 2\pi\sqrt{\frac{l}{g}}$

   $\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$ → Divide both sides by 2π.

   $\left(\frac{T}{2\pi}\right)^2 = \sqrt{\left(\frac{l}{g}\right)^2}$ → Square both sides.

   $\frac{T^2}{4\pi^2} = \frac{l}{g}$

   $l = \frac{T^2 g}{4\pi^2}$ → Multiply both sides by g.



A pendulum, where l is its length

EXERCISE 10

In each of the questions below, make the letter in brackets the subject of the formula.

1.	$ax - kx = 3$	(x)
2.	$d = s \times t$	(t)
3.	$px^2 - qx^2 = 4p - 4q,\ p \neq q$	(x)
4.	$W = fd$	(f)
5.	$F = qvB$	(B)
6.	$W = I^2Rt\ (I > 0)$	(I)
7.	$V = \pi r^2 h$	(r)
8.	$mx^2 - nx^2 = t$	(x)
9.	$n = \frac{m}{M}$	(m)
10.	$V = \pi r^2 h$	(h)

11. $P = \frac{w}{t}$ (t)

12. $E = \frac{V}{d}$ (d)

13. $mx^2 - nx^2 = n^2 - m^2,\ m \neq n$ (x)

14. $ax - a - bx + b = 0,\ a \neq b$ (x)

15. $pv = nRT$ (R)

16. $T = \frac{1}{v}$ (v)

17. $P = 2(l + b)$ (b)

18. $I = \frac{V}{R}$ (R)

19. $D = \frac{m}{v}$ (v)

20. $P = \frac{f}{a}$ (f)

21. $c = \frac{m}{MV}$ (M)

22. $kx - k = x^2 - 1$ (x)

23. $W = \frac{V^2}{R}t\ (V > 0)$ (V)

EXERCISE 11

1. Consider the formula $V = u + at$.
   a) Make a the subject of the formula.
   b) Hence determine the value of a, if $V = 75$, $u = 15$ and $t = 5$.

2. Consider the formula $\frac{1}{s} + \frac{1}{t} = \frac{1}{w}$.
   a) Make t the subject of the formula.
   b) Hence determine the value of t, if $s = 3{,}75$ and $w = 1{,}25$.

3. Consider the formula $A = P(1 + in)$.
   a) Make i the subject of the formula.
   b) Hence determine the value of i, if $A =$ R13 000, $P =$ R10 000 and $n = 3$.

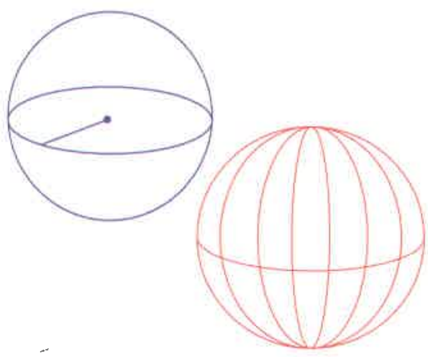
A cylinder

4. Consider the formula $A = P(1 + i)^n$.
   a) Make i the subject of the formula.
   b) Hence determine (correct to 3 decimal places) the value of i, if $P =$ R13 000, $n = 6$ years and $A =$ R24 983,42.

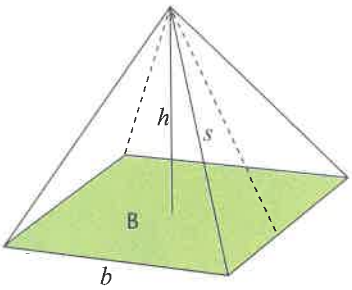
5. Consider the formula $S_n = \frac{n}{2}(2a + (n - 1)d)$.
   a) Make d the subject of the formula.
   b) Hence determine the value of d, if $S_n = 5\ 150$, $a = 5$ and $n = 50$.

6. The surface area of a right cylinder is given by the formula $A = 2\pi r(r + h)$.
   a) Make h the subject of the formula.
   b) Hence determine the value of h, if $A = 520\pi$ units^2 and $r = 13$ units.

7. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.
   a) Make r the subject of the formula.
   b) Hence determine (correct to 2 decimal places) the value of r, if $V = 4\ 188{,}79$ units^3.



A sphere

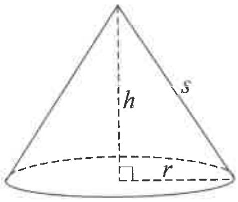


A pyramid

8. A pyramid on a square base has a:
- total surface area given by $A = b^2 + 2bs$.
- volume given by $V = \frac{1}{3}b^2h$.

Determine the:

a) height (h) of the pyramid in terms of V and b
b) side of the base (b) in terms of V and h
c) slant height (s) in terms of A and b.



A cone

9. A cone with radius (r), height (h) and slant height (s) has a:
- volume given by $V = \frac{1}{3}\pi r^2h$
- total surface area given by $A = \pi r(r + s)$
- slant height given by $s^2 = h^2 + r^2$.

Use the formulae to determine:

a) h (the height of the cone)
b) r (the radius of the circular base) in terms of V and h
c) s (the slant height)
d) r (the radius of the circle), if $s = 50$ cm and $h = 48$ cm.

Unit 5: Solve linear inequalities

When you solve equations, you can expect to find one (or more) specific solutions.
$2x - 10 = 0$ is a linear equation with exactly 1 solution, $x = 5$.
$x^2 - 5x - 6 = 0$ is a quadratic equation with 2 solutions, $x = -1$ or $x = 6$.

Inequalities often have an infinite number of solutions. This makes it impossible to state each solution individually.

$x + 7 > 10, x \in \mathbb{R}$
$\therefore x > 3$

This is a linear inequality with an infinite number of solutions.
x can be any real number greater in value than 3.

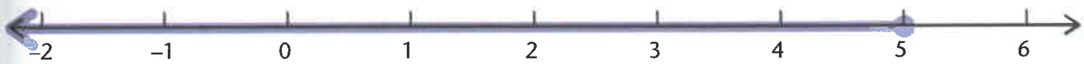
Shown on a number line, the solution is:

The solid red line indicates all the numbers greater than 3. The open circle on 3 shows $x = 3$ is not a solution. 3,24 and irrational numbers, such as $\sqrt{11}$ and 4,235… are solutions to $x > 3$.

$x + 8 \leq 13, x \in \mathbb{R}$
$\therefore x \leq 5$

This is a linear inequality with an infinite number of solutions.
x can be any real number less than or equal to 5.

Shown on a number line, the solution is:



5 has a solid circle because $x = 5$ is a solution.
$\sqrt{7}; -\sqrt{13}; 0{,}47…; \frac{2}{3}$ are a few of the possible solutions to $x \leq 5$.

Inequalities are affected by multiplication (or division) of negative numbers.

Consider $-3 < 2 < 5$. Do you see that, if you multiply (or divide) by –1, without switching the inequality signs, you get $3 < 2 < -5$. This makes no sense as $3 \geq 2$ and $2 \geq -5$, so you must switch the signs, which gives you $3 > 2 > -5$.

Adding (or subtracting) equal amounts on both sides maintains the relationship:

$-4 < 2$ and adding 4 to both sides gives you $-4 + 4 < 2 + 4$, thus $0 < 6$.

$-4 < 2$ and subtracting 5 from both sides gives you $-4 - 5 < 2 - 5$, thus $-9 < -3$.

KEY WORDS

real numbers, $x \in \mathbb{R}$:
- x can be rational or irrational
- x cannot be undefined
- x cannot be imaginary.

integers, $x \in \mathbb{Z}$:
$x \in \{…; -2; -1; 0; 1; 2; …\}$
natural numbers, $x \in \mathbb{N}$:
$x \in \{1; 2; 3; …\}$
whole numbers, $x \in \mathbb{N}^0$
$x \in \{0; 1; 2; 3; …\}$

WORKED EXAMPLES

For each of the questions below:

a) Solve for x.

b) Show your answer on a number line.

1. $3x - 5 < 2x + 1$ and $x \in \mathbb{R}$
2. $3x - 5 < 2x + 1$ and $x \in \mathbb{Z}$
3. $3x - 5 < 2x + 1$ and $x \in \mathbb{N}$

SOLUTIONS

1. a) $3x - 5 < 2x + 1$

$3x - 5 + 5 - 2x < 2x + 1 + 5 - 2x$

$\therefore x < 6$

1. b) 

2. a) $3x - 5 < 2x + 1$

$3x - 5 + 5 - 2x < 2x + 1 + 5 - 2x$

$\therefore x < 6$ and $x \in \{5; 4; 3; \ldots\}$

2. b) 

3. a) $3x - 5 < 2x + 1$

$3x - 5 + 5 - 2x < 2x + 1 + 5 - 2x$

$\therefore x < 6$ and $x \in \{1; 2; 3; 4; 5\}$

3. b) 

KEY CONCEPTS

If x is any real number:
$x \in \mathbb{R}$
You cannot list the solutions.
The number line will be shaded with a solid line.
If x is a natural number:
$x \in \mathbb{N}$
You list the numbers.
You use dots on the number line.

If x is a whole number:
$x \in \mathbb{N}_0$
You list the numbers.
You use dots on the number line.
If x is an integer:
$x \in \mathbb{Z}$
You list the numbers.
You use dots on the number line.

EXERCISE 12

Solve for x and show your answer on a number line:

1. $6x - 11 < 2x + 7$ and $x \in \mathbb{R}$
2. $x - 4 \geq 3x + 2$ and $x \in \mathbb{R}$
3. $4 - 2x \leq x - 2$ and $x \in \mathbb{Z}$
4. $3x + 2 > 5x + 7$ and $x \in \mathbb{N}$
5. $4x - 3 \leq 2x - 11$ and $x \in \mathbb{R}$
6. $7x - 2(3 - 2x) > 5(x - 3)$ and $x \in \mathbb{N}$
7. $2(4 - 3x) \geq 20$ and $x \in \mathbb{R}$
8. $5x - 2(x - 3) < x + 2$ and $x \in \mathbb{Z}$

Inequalities with fractions

It is easier to multiply each value by the LCM of the denominators.

You switch the inequality sign, if you multiply or divide by a negative.

WORKED EXAMPLES

Solve for x, if $\frac{x-5}{2} - \frac{5x-13}{6} > \frac{3x+5}{4} - 7$ and $x \in \mathbb{R}$. Show your answer on a number line.

SOLUTION

$\frac{x-5}{2} - \frac{5x-13}{6} > \frac{3x+5}{4} - 7$

The LCM of the denominators is 12.

$$\frac{12}{1} \times \frac{x-5}{2} - \frac{12}{1} \times \frac{5x-13}{6} > \frac{12}{1} \times \frac{3x+5}{4} - \frac{12}{1} \times \frac{7}{1}$$

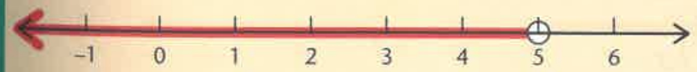
$$6(x - 5) - 2(5x - 13) > 3(3x + 5) - 84$$

$$6x - 30 - 10x + 26 > 9x + 15 - 84$$

$$6x - 10x - 9x > 15 - 84 + 30 - 26$$

$$\therefore -13x > -65$$

$$\therefore x < 5$$



EXERCISE 13

Solve for x and show your answer on a number line.

1. $x - \frac{x-3}{2} \geq 2{,}5 + \frac{5x}{6}$ and $x \in \mathbb{Z}$
2. $\frac{3x+1}{5} - \frac{x}{3} \geq \frac{x+12}{15}$ and $x \in \mathbb{N}$
3. $\frac{3x-2}{4} - \frac{6-x}{3} < 4$ and $x \in \mathbb{R}$
4. $\frac{3x-1}{2} - \frac{3-x}{3} < \frac{x}{12} + 2$ and $x \in \mathbb{R}$

You can also solve inequalities with two inequality signs.

$-2 < x \leq 5$ tells you that x lies between -2 and 5. x cannot be -2, but it can equal 5. On a number line there will be restrictions on both sides.

WORKED EXAMPLES

Solve for x and show your answer on a number line, if:

1. $-16 < 2 - 6x \leq 22$ and $x \in \mathbb{R}$
2. $-12 < 4 - 2x \leq 6$ and $x \in \mathbb{N}^0$

SOLUTIONS

1. $-16 - 2 < 2 - 2 - 6x \leq 32 - 2$
   $\therefore -18 < -6x \leq 30$ and $+18 > +6x \geq -30$
   $\therefore 3 > x \geq -5$
   $\therefore -5 \leq x < 3$

2. $-12 - 4 < 4 - 4 - 2x \leq 6 - 4$
   $\therefore -16 < -2x \leq 2$ and $+16 > +2x \geq -2$
   $\therefore 8 > x \geq -1$
   $\therefore -1 \leq x < 8$ and $x \in \{0; 1; 2; \ldots; 7\}$

EXERCISE 14

In each of the inequalities below, solve for x and show the solution on a number line.

1. $-15 \leq 4x + 1 < 1$ and $x \in \mathbb{R}$
2. $-15 < 1 - 4x < 5$ and $x \in \mathbb{R}$
3. $5 < 2x + 3 \leq 17$ and $x \in \mathbb{R}$
4. $-5 < 2x - 3 \leq 11$ and $x \in \mathbb{Z}$
5. $-12 \leq 3 - 3x < 6$ and $x \in \mathbb{N}$
6. $-\frac{1}{2} < \frac{x-1}{6} \leq \frac{1}{12}$ and $x \in \mathbb{Z}$
7. $-2 \leq 2 - 4x < 6$ and $x \in \mathbb{R}$
8. $0 < \frac{x}{3} + 1 \leq 3$ and $x \in \mathbb{R}$
9. $\frac{1}{4} \leq \frac{x}{12} + \frac{1}{3} < 1\frac{1}{6}$ and $x \in \mathbb{Z}$

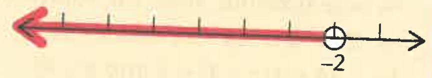
Interval notation is a very useful way to write the solution of an inequality, but can only be used if you are working with real numbers.

$-5 < x \leq 4$, $x \in \mathbb{R} \therefore x \in (-5;4]$
And $x \geq 2$, $x \in \mathbb{R} \therefore x \in (2;\infty)$

WORKED EXAMPLES

Represent each number line below in both inequality notation and interval notation.

1. 

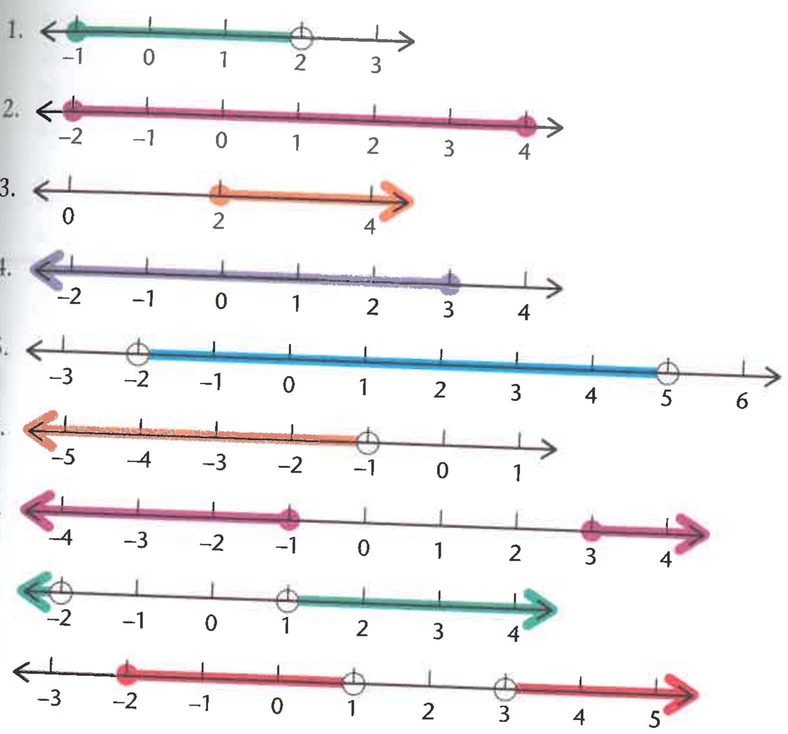
2. 

SOLUTIONS

1. inequality notation: $-3 < x \leq 4$, $x \in \mathbb{R}$ — interval notation: $x \in (-3;4]$
2. inequality notation: $x < 2$, $x \in \mathbb{R}$ — interval notation: $(-\infty;2)$

EXERCISE 15

Represent each of the number lines below in both interval notation and in inequality notation.



KEY CONCEPTS

Interval notation can only be used if you are working with real numbers.
Non-included values:
< or > in inequality notation
○ on a number line
(or) in interval notation

Infinity signs (∞ and $-\infty$) always have round brackets because endpoints are always excluded.
Included values:
≤ or ≥ in inequality notation
● on a number line
[or] in interval notation

EXERCISE 16

1. For each of the questions below:
   Solve for x and show your answer on a number line.
   Where possible, state the solution in interval notation.
   a) $2x - 5 \leq 3(x - 2)$ and $x \in \mathbb{R}$
   b) $-24 \leq 3(x - 4) < 6$ and $x \in \mathbb{R}$
   c) $2(3 - 2x) + 3 > 3(2x - 5) - 2x$ and $x \in \mathbb{N}_0$
   d) $-1 < \frac{1}{3}(x - 2) \leq 3$ and $x \in \mathbb{Z}$
   e) $-3\frac{4}{9} \leq \frac{5}{3}x + \frac{1}{9} \leq 1\frac{7}{9}$ and $x \in \mathbb{R}$

2. Represent the number lines below in both interval notation and inequality notation.

a) 

b) 

c) 

d) 

Revision

1. Solve for x.
   a) $5x^2 = x$ (3)
   b) $4x(2x - 5) = 28$ (4)
   c) $12 = x(11 - 2x)$ (4)
   d) $\frac{1}{2}(3x + 5) - \frac{3}{4}(5x - 2) = 6$ (5)
   e) $5x^2 + 30x = 80$ (3)
   f) $\frac{x(x-2)}{6} - \frac{1x}{3} = 1 - \frac{1x}{4}$ (6)
   g) $ax - bx = b^2 - a^2 \ (a \neq b)$ (3)
   h) $d^2 + dx = h^2 + hx \ (d \neq h)$ (5)
   i) $\frac{1}{x} = \frac{1}{w} - \frac{1}{v}$ (4)
   j) $9x^2(x^2 - 1) - 4(x^2 - 1) = 0$ (6)

2. Given $3 - 4x \geq x + 5$ and $x \in \mathbb{R}$.
   a) Solve for x. (3)
   b) Show your answer on a number line. (2)
   c) Write the solution in interval notation. (2)

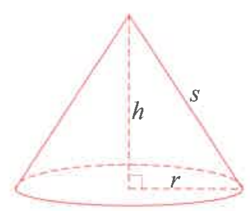
3. Given $3(2x - 4) - x \geq 2(x - 5) - 17$ and $x \in \mathbb{Z}$.
   a) Solve for x. (4)
   b) Illustrate your solution on a number line. (6)
   c) Explain why you can't write this solution in interval notation. (1)

4. Given $\frac{3}{4} < \frac{x+1}{2} \leq \frac{7}{3}$ and $x \in \mathbb{R}$.
   a) Solve for x. (4)
   b) Show your answer on a number line. (2)
   c) Write the solution in interval notation. (2)

5. Given $2y + 3x = -6$ and $4y - 3x = 24$.
   a) Draw the graphs of the equations on the same system of axes.
      Write down the coordinates of the point of intersection of these graphs. (5)
   b) Solve simultaneously for x and y (using algebraic methods) and show your calculations. (6)

6. Two natural numbers have a difference of 13. The sum of their squares is 265. What are the two numbers? (6)

7. Given $r = \sqrt{\frac{V}{\pi h}}$.
   a) Make h the subject of the formula. (3)
   b) Solve for h, if $V = 400$ and $r = 5$ (correct to 2 decimal places). (2)



8. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. Determine:
   a) r (in terms of V and π) (3)
   b) the length of the radius (correct to 2 decimal places), if the volume of the sphere is 152 895 cm^3. (2)

9. Solve for x:
   a) $8x^3 = 27$ (2)
   b) $3 \times 2^{x-1} = 24$ (3)
   c) $5 \times 9^x = 405$ (3)
   d) $3^{2x} - 12 \times 3^x + 27 = 0$ (4)

10. Consider the equation $k = \frac{x^2 + 4x - 21}{x^2 - 9x + 18}$.
    a) For which value(s) of x is k undefined? (3)
    b) Solve for x, if $k = 0$. (3)

11. Complete the following.
    a) Solve for k, if $k^2 - 10k + 16 = 0$. (3)
    b) Hence solve for x, if $4^x - 10 \times 2^x + 16 = 0$. (5)

12. Two numbers have a sum of 20 and a product of 64.
    What are the two numbers? (5)

13. The sum of the squares of two consecutive odd numbers is 1 154.
    What are the two numbers? (5)

14. Towns A and B are 540 km apart. Poloko left Town A at 7:00 and travelled towards Town B. Thomas left Town B at 8:00 and travelled towards Town A. Poloko and Thomas passed each other 300 km from Town A. Thomas travelled 20 km/h faster than Poloko. At what time did they meet? (6)

15. An aeroplane can carry 40 passengers in total. First class passengers are allowed 25 kg of luggage. Economy class passengers are allowed 20 kg of luggage. The maximum combined weight for the luggage of all the passengers is 920 kg. What is the maximum number of first class passengers that the aeroplane can carry? (6)