Topic 8

Algebraic expressions

Maths ideas

- Identify coefficients and exponents in algebraic expressions.
- Add and subtract like terms.
- Simplify algebraic expressions.
- Multiply integers and monomials by polynomials.
- Divide polynomials by integers or monomials.
- Find the product of two binomials.
- Find the square of a binomial.

Key words

- **expression** – a group of terms separated by + or – signs without an = sign
- **symbol** – something that stands for something else; for example, '+' is the symbol for 'add'
- **variable** – a letter of the alphabet that we use to represent an unknown value

Expand and simplify algebraic expressions

We often use expressions with numbers and symbols to describe different situations. An **expression** is a group of terms separated by + and – signs. **Symbols** are signs we use in Mathematics, for example +, –, × and ÷ to describe mathematical operations.

Terms can be a single number (a constant), or numbers and **variables** multiplied together. These mathematical expressions are called algebraic expressions when they contain letters in the place of unknown values. In algebra, we often use the letters x and y in the place of an unknown value, but any letter may be used. We call these letters variables because they can stand for any unknown number.

As the variable in an expression takes on different numerical values, the value of the whole expression changes. The expression $11x^2 - 3x + 4$ has a value of 12 when $x = 1$ but has a value of 4 when $x = 0$.

A numerical term containing no variable is called a **constant**, as its value does not change. $5x - 10 + x - 8$ is an expression that may be simplified to $6x - 18$. In this simplified expression, the variable term is $6x$ and the constant is –18.
The number in front of the variable is the coefficient and in this case, the coefficient of the variable term is 6.

You will study equations in the next topic.

We use certain conventions when using letters in the place of numbers:

- **We don't always use a multiplication (×) or a divide (÷) sign in algebra.** Numbers and letters are written next to each other, which indicates multiplication. For example, $5abc$ means $5 \times a \times b \times c$. Numbers and letters are written in fraction form to indicate division. For example $\frac{7a^2bc}{2ab}$ means $(7a^2bc) \div (2ab)$
- **We always write numbers in front of letters.** For example $7 \times a \times b \times c$ is written $7abc$ and not $abc7$.
- **Letters are written in alphabetical order.** $10 \times a \times b \times c$ is written $10abc$ and not $10cba$ or $10cab$ or $10bca$.
- **The order in which the letters (and numbers) are multiplied doesn't matter.** You have already learnt that $4 \times 2 \times 1 \times 3 = 24$ no matter what order the numbers are written or multiplied.
- **In algebra, we don't write a^1 or $1a$.** We simply write a. The superscript 1 and the coefficient 1 are unnecessary, because anything multiplied by 1 or raised to the power of 1 doesn't change the value.

Add and subtract like terms in polynomials

In algebraic expressions, only terms which that identical variables may be added or subtracted. $3xy^2$ and $6x^2y$ are unlike terms as the exponents of the variables are not identical. The terms $-5abc^2$ and $7abc^2$ are like terms and these terms may be added: $-5abc^2 + 7abc^2 = 2abc^2$

You can add and subtract like terms, but you cannot add and subtract unlike terms.

First get rid of any brackets when adding or subtracting **polynomials**.

When you subtract $(2x - 5y)$ **FROM** $(6x - 3y)$ always write the part that follows **FROM** first and then be careful of the brackets and the signs. Remember your integer rules for addition and subtraction:

$(6x - 3y) - (2x - 5y) = 6x - 3y - 2x + 5y$ ← two negatives multiplied together give a positive answer

This expression can be simplified to $4x + 2y$

$4x$ and $2y$ are unlike terms and the expression cannot be simplified any further.

An expression where all the exponents are whole numbers is called a polynomial.

$2y^2 - 4y - y^3$ is a polynomial but $12x^{-2} - 5y^{\frac{1}{2}}$ is not a polynomial.

The **degree of a polynomial** is the highest power of the variable. In $2y^2 - 4y - y^3$, the degree of the polynomial is 3.

Polynomials with only one, two or three terms have special names:

A **monomial** is an expression containing only one term, for example $6a^2$ or $-2(3pq)^2$.

A **binomial** is an expression containing two terms, for example $x^2 - 4y^3$ or $(a + b) + 3(c - d)$. Note that each bracket is part of a term.

A **trinomial** has three terms, for example $2a^2 - 3b + 4c$

Example

1. Add $(7a^2 + 4a - 2)$ and $(-3a^2 + 6a - 10)$
2. Subtract $(4x^2 - 2x + 3)$ from $(5x^2 + 3x - 6)$
3. In the expression $-5x^4 + 3x^2 - 12x - 8$
   a) How many terms are in the expression?
   b) The constant term is …
   c) The degree of the expression is …
   d) What is the coefficient of the x^2 term?
   e) What is the value of the expression, if $x = -1$?

Answers

1. $(7a^2 + 4a - 2) + (-3a^2 + 6a - 10)$
   $= 7a^2 + 4a - 2 - 3a^2 + 6a - 10$ ← get rid of the brackets
   $= 7a^2 - 3a^2 + 4a + 6a - 2 - 10$ ← simplify by adding like terms
   $= 4a^2 + 10a - 12$ ← simplify

2. $(5x^2 + 3x - 6) - (4x^2 - 2x + 3)$ ← write the expression that follows "from" first
   $= 5x^2 + 3x - 6 - 4x^2 + 2x - 3$ ← multiply out the brackets
   $= 5x^2 - 4x^2 + 3x + 2x - 6 - 3$ ← group like terms together
   $= x^2 + 5x - 9$ ← simplify by adding like terms

3. a) There are 4 terms in the expression ← terms are separated by + or – signs
   b) The constant term is –8 ← this term has no variables
   c) The degree of the polynomial is 4 ← the highest exponent
   d) The coefficient of the x^2 term = 3 ← the numerical portion of the term.
   e) $-5(-1)^4 + 3(-1)^2 - 12(-1) - 8 = 2$ ← substitute $x = -1$ into the expression

Key words

- **constant** – a number in an expression or equation that is independent of any variable and doesn't change
- **polynomial** – an expression consisting of many terms
- **degree of a polynomial** – the highest power of the variable
- **monomial** – an expression containing only one term
- **binomial** – an expression containing two terms
- **trinomial** – an expression containing three terms

EXERCISE 8.1

1. Write down whether these expressions are monomials, binomials or trinomials:
   a) $2x^2 + 11y$
   b) $17abc - 28bdp + 122fgh$
   c) $23ab - 14ab^2$
   d) $(27m^2n)^3$

2. Find the value of $4x^2 + 10x - 17$ if:
   a) $x = 0$
   b) $x = 6$
   c) $x = -4$

3. In the expression $12x^2 + 2x - 15$:
   a) How many terms are in the expression?
   b) Write down the coefficient of x^2.
   c) What is the value of the constant term?
   d) Calculate the value of the expression, when $x = -1$.
   e) Write down the degree of the expression.

4. a) Are $3x^2y^3$ and $-5x^3y^2$ like terms?
   b) Are $4a^2b^3$ and $-2b^3a^2$ like terms?

5. Simplify if possible:
   a) $7x^2 - 14x^2$
   b) $15abc + 12bca - 13acb$
   c) $8xy^2 + 7x^2y$
   d) $100x^2y^3 - 50x^2y^3$

6. Add the following polynomials:
   a) $(2x^2 - 4x + 3) + (x^2 + 4x - 4)$
   b) $(-x^2 + 11x - 12) + (x^2 - 13x - 15)$
   c) $(13x^2 - 10) + (2x^2 - 11x + 5)$
   d) $(7x^2 - 8x) + (x^2 - 5) + (2x^2 + 8x + 5)$

7. What must be added to $5x^2 - 3x + 8$ to give $7x^2 - 3x + 10$?

8. a) Subtract $(4x^2 - 2xy + 6y^2)$ from $(3x^2 + 3xy - 6y^2)$
   b) From $9a^2 + 4a - 10$ subtract $6a^2 + 3a - 9$

Multiply and divide polynomials

When you have a polynomial in a bracket multiplied by an integer or a monomial, multiply every term inside the bracket by the integer or monomial in front of the bracket. Remember the rules of multiplying negative numbers as well as the exponent laws:

For example $3(x^2 - 4x) = 3x^2 - 12x$

and $-x(4x^2 + 2x - 1) = -4x^3 - 2x^2 + x$

When you divide a polynomial by an integer or a monomial, separate the terms and then cancel where possible. You may NOT cancel across terms.

For example: $\frac{5x^2 + 1}{x^2} = \frac{5x^2}{x^2} + \frac{1}{x^2} = 5 + \frac{1}{x^2}$ $(x \neq 0)$. This expression may not be simplified further.

Remember that the denominator of a fraction may never equal zero.

Multiply out brackets before adding like terms when simplifying expressions:

For example: $2a(a - 6b + 3c) - 5a(4a - b + 7c)$

$= 2a^2 - 12ab + 6ac - 20a^2 + 5ab - 35ac$ ← multiply out the brackets

$= -18a^2 - 7ab - 29ac$ ← simplify like terms

Simplify:

a) $2(4a + 5b) - 3(a - 3b) - 8(2b)$ b) $\frac{3x^2 - 6x + 12}{4x}$ $(x \neq 0)$

Answers

a) $8a + 10b - 3a + 9b - 16b$ ← multiply out the brackets

$= 8a - 3a + 10b + 9b - 16b$ ← group like terms together

$= 5a + 3b$ ← simplify like terms

b) $\frac{3x^2 - 6x + 12}{4x} = \frac{3x^2}{4x} - \frac{6x}{4x} + \frac{12}{4x}$ ← separate the terms

$= \frac{3}{4}x - \frac{3}{2} + \frac{3}{x}$ ← simplify each term

EXERCISE 8.2

Simplify the following expressions:

1. a) $6 + 3(x - 2)$ b) $5x - 2x(x + 2)$

2. a) $5(2a - b) + 2(3b - 4a)$ b) $2(x + y) + 4(3x - 2y) - 5(2x - 3y)$
c) $4a(a + b) - 2b(3a - 5b) + 2ab$

3. a) $7(x^2 - x + 2) + (6x - 14)$ b) $2(3x^2 + 6x - 1) - (2x - 4)$
c) $-3(x^3 + 2x^2 - x) - x^2(2x + 1)$

4. a) $\frac{a^2 - 4a}{a}$ $(a \neq 0)$ b) $\frac{10a^2 - 5a}{5a}$ $(a \neq 0)$

5. a) $\frac{10x^2 - 4x + 9}{x}$ $(x \neq 0)$ b) $\frac{9x^4 - 6x^3 - 3x^2 + 12x}{3x}$ $(x \neq 0)$
c) $\frac{x^4 - 8x^3 + x^2 - 6}{4x^2}$ $(x \neq 0)$ d) $\frac{8x^4 - 6x(-x)(2x^2)}{-x}$ $(x \neq 0)$

6. a) $\sqrt{49x^2}$ b) $(-2x^2)^3 + (3x^3)^2$
c) $(\frac{2xy}{8xy})^3$ d) $2\sqrt[3]{x^3} - \sqrt{64x^2} + \sqrt{36x^4 - 11x^4}$

Multiply binomials

When multiplying binomials, multiply every term in the first bracket with every term in the second bracket. Then simplify where possible.

$(6x + 4)(3x + 5) = 6x(3x + 5) + 4(3x + 5)$ ← multiply each term in the first bracket with every term in the 2nd bracket

$= 18x^2 + 30x + 12x + 20$ ← multiply out the brackets

$= 18x^2 + 42x + 20$ ← add the like terms

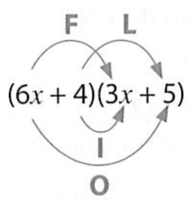
You can get the same result by using the acronym FOIL to multiply out two binomials.

F = multiply first terms together

O = multiply outer terms together

I = multiply inner terms together

L = multiply last terms together



For example: Multiply $(6x + 4)(3x + 5)$

$(6x + 4)(3x + 5)$

$= 18x^2 + 30x + 12x + 20$

$= 18x^2 + 42x + 20$

When you have a square of a binomial, write out both brackets and then use FOIL to find the product.

For example, $(2x + 3)^2 = (2x + 3)(2x + 3)$

Use FOIL to find the product: $4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$

The **middle 2 terms** are identical and the **first** and **last** terms are perfect squares.

The first and last terms are always positive as they are perfect squares. The middle term of the trinomial takes the sign inside the bracket:

$(2x + 3)^2 = 4x^2 + 12x + 9$ but $(2x - 3)^2 = 4x^2 - 12x + 9$

The square of any binomial in the form $(x + y)^2$ expands into a trinomial of the type: $x^2 + 2xy + y^2$

When you multiply two binomials together and they are identical except for the sign in the bracket, the middle two terms will cancel each other out and the result will be a binomial difference of squares.

$(x - y)(x + y) = x^2 + xy - xy - y^2$ ← multiply each term using FOIL

$= x^2 - y^2$ ← middle terms cancel

The signs in the brackets are different, so the last term of the product is always negative. This is why the product is called the difference of two squares. Both terms are squares and there is a minus sign between them.

Example

Find the product:

$(4x - 5)^2$

Answer

$(4x - 5)^2 = (4x - 5)(4x - 5)$ ← write the bracket out twice

$= 16x^2 - 20x - 20x + 25$ ← multiply out the brackets using FOIL

$= 16x^2 - 40x + 25$ ← combine the like terms

EXERCISE 8.3

Simplify:

1. a) $(x + 2)(x - 4)$ b) $(2x + 1)(3x - 5)$
   c) $(x + y)(x + 2y)$ d) $(3x - 4y)(2x - 5y)$
2. a) $(x - 4)^2$ b) $(3x + 2)^2$
   c) $(6x - 5y)^2$ d) $(7x + 4y)^2$
3. a) $(x - 2)(x + 2)$ b) $(2x + 1)(2x - 1)$
   c) $(x + 3y)(x - 3y)$ d) $(7 - y)(7 + y)$
4. a) $(2a - 3b)(4c + 5d)$ b) $(9y - 7)(7 + 9y)$
   c) $(\frac{1}{2}x - 1)^2$ d) $(0{,}6p + 1)(0{,}6p - 1)$
5. a) $(x + 3)^2 + (2x + 1)^2$ b) $(x - 3)^2 - (x + 2)^2$
   c) $(2a - 3b)^2 - (2a + 3b)^2$ d) $(7x - 12)^2 - 144$
6. A square has sides of length $(2a - b)$ metres. Write out a formula for its area.

Challenge

Simplify:

$$\frac{\sqrt{36x^4y^2}\,\sqrt[3]{125x^6y^6}\,\sqrt{144x^2y^4}}{\sqrt[3]{64x^9y^9}\,\sqrt{81x^2y^2}\,\sqrt[3]{1\,000x^3y^3}}$$

Revision

1. Find the value of $-3x^2 + 8x - 15$ if $x = -2$. (2)

2. In the expression $16x^3 - 9x^2 + 10x - 1$
   a) How many terms are in the expression?
   b) Write down the coefficient of x^2
   c) What is the value of the constant term?
   d) Calculate the value of the expression when $x = 3$.
   e) Write down the degree of the expression. (5)

3. Simplify, if possible:
   a) $48x^2y^3 - 12x^2y^3$ (2)
   b) $36x + 28y - 19x + 18y$ (2)

4. Add the following polynomials:
   a) $(-5x^2 + 8) + (2x^2 - 25x - 9)$ (3)
   b) $(16x^2 - 8x + 2) + (3x^2 - 5x) + (2x^2 + 12x - 3)$ (3)

5. Subtract $(6x^2 - 5xy + 10y^2)$ from $(7x^2 - xy + 4y^2)$. (3)

6. Simplify the following expressions:
   a) $3(2a - 5b) + 2(5b - 7a)$ (4)
   b) $2(3x^2 + 6x - 1) - (2x - 4)$ (5)
   c) $\frac{-2x^2 + 3x - 5}{x}$ $(x \neq 0)$ (3)
   d) $(-3x^2)^3 + (4x^3)^2$ (3)
   e) $\frac{(4xy)^3}{(8x^2y)^2}$ (3)

7. Expand and simplify:
   a) $(3x + 2)(2x - 5)$ (3)
   b) $(2x - 3)^2$ (3)
   c) $(7x - 1)(7x + 1)$ (2)

8. Expand and simplify:
   a) $(x + 1)^2 + (2x + 3)^2$ (5)
   b) $(x - 3)^2 - 9$ (3)

9. Simplify: $(x + 2)^2 - (x - 2)^2$ (3)

10. Write out a formula for the area of a rectangle with length $(2x + 1)$ m and breadth $(x - 3)$ m. (3)

Total: 60 marks