Topic 2

Integers

Maths ideas

- Calculate with integers.
- Revise properties of integers.
- Calculate with squares, cubes, square roots and cube roots.

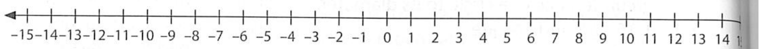
Key words

- **additive inverse** – the number you add to another number to get 0, for example –18 and 18 are additive inverses
- **positive integers** – whole numbers greater than 0
- **negative integers** – whole numbers less than 0

Calculations with integers

Every whole number has an opposite number called its **additive inverse**. A number and its additive inverse are the same distance away from zero. If you look at the number line below, you can see that every number has an opposite, except for zero.

Whole numbers greater than 0 are called **positive integers** and whole numbers less than 0 are called **negative integers**. The integer 0 is neutral. It is neither negative nor positive. We can call integers the set of whole numbers and their opposites. They can be illustrated on a number line with the negative integers to the left of zero and the positive integers to the right of zero.



Negative integers are always written with a negative sign. You read the negative sign as 'minus', for example you would read –14 as *minus* 14.

Positive integers are usually written without a positive sign. If a positive sign is written, you read the positive sign as 'plus', for example you would read +15 as *plus* 15.

Integers can be used in real-life situations, for example rising and falling temperatures, earning and spending money and stock market gains and losses.

Above sea level is the opposite of below sea level. Other opposites used in real-life problems are shown in the table.

positive	negative
increase	decrease
top	bottom
ascending	descending
forward	backward
right	left
smallest	largest

The mathematical symbol < means 'is less than' and > means 'is greater than'.

When you need to compare two integers or to arrange several integers in order, think of their relative positions on the number line.

The further to the *right* a number is on the number line, the *greater* its value.

The further to the *left* a number is on the number line, the *smaller* its value.

When we list numbers from smallest to largest we write the numbers in ascending order. When we write the largest number first and then the numbers from largest to smallest we write the numbers in descending order.

Example

1. Which number is larger, –120 or –102?
2. Arrange the following numbers in descending order:
   –126; 35; –215; 99; 0; –220

Answers

1. –102 > –120 ← –102 is further to the right on the number line
2. 99; 35; 0; –126; –215; –220 ← write the numbers from largest to smallest

EXERCISE 2.1

1. Which number is greater in these pairs?
   a) –716 405 or –393 910
   b) –11 050 or –22 110
2. Which number is smaller in these pairs?
   a) –78 319 or –79 361
   b) 0 or –1 010
3. Arrange these numbers in ascending order:
   a) 10 °C; –9 °C; 8 °C; –12 °C; 0 °C
   b) R305; –R36; R192; –R70
4. Arrange these numbers in descending order:
   a) –17 °C; –19 °C; 0 °C; –2 °C; 1 °C
   b) –R51 270; –R68 160; –R109 102; –R71 450
5. Calculate the value of the expression $3x^2 + 4x - 5$ when
   a) $x = -1$ b) $x = -2$ c) $x = 0$
6. The temperature in Moscow was 2 °C. Then it dropped to 8 °C below zero. By how many degrees did the temperature drop?
7. Which temperature is lower?
   a) –9 °C or –10 °C
   b) –25 °C or –52 °C
8. Mount Everest is 8 848 m above sea level and the Dead Sea is 400 m below sea level. What is the difference between the two elevations?
9. Mercury has a melting point of –39 °C and the freezing point of methanol is –98 °C. How much warmer is the melting point of mercury than the freezing point of methanol?
10. The Dow Jones average (a stock market share index) dropped from 12 837 to 12 503 in one week in June 2012.
    a) Calculate the drop in the share index.
    b) If the price continued to drop at the same rate, calculate the Dow Jones average after 4 more weeks.

Did you know?

15 135 120 can be divided exactly by any number from 1 to 16.

Challenge

Subtract the sum of the ninth to the sixteenth prime numbers from the sum of the first eight prime numbers.

Key words

- **like terms** – terms that are identical in respect of their variables
- **coefficient** – a number in front of a variable

Properties of integers

Add and subtract terms with integer coefficients

You may only add or subtract **like terms** when simplifying an expression or solving an equation. Like terms have exactly the same variables and exponents. The number part of a term is called the **coefficient**, for example $-5x^2y$ and $2x^2y$ have integer coefficients −5 and 2 and the variable portion of each term is x^2y. They are like terms and may be added

$-5x^2y + 2x^2y = -3x^2y$. The coefficients of the variables are added. The variable part of each term remains unchanged.

$-4xy^2$ and $-4x^2y$ are not like terms as the variable part of the terms differ. These terms may not be added or subtracted.

The sum of two positive like terms is positive, for example
$+12xy + 34xy = +46xy$

When numbers are positive we do not have to write the '+' sign: $12xy + 34xy = 46xy$

The sum of two negative like terms is negative, for example
$(-22abc) + (-34abc) = -56abc$

We can also write this without the brackets as $-22abc - 34abc = -56abc$

To find the sum of two like terms with different signs, find the difference between the coefficients. The answer will have the sign of the coefficient with the greater numerical value (without the + or − signs), for example: $-48x^2 + 17x^2 = -31x^2$ but $48x^2 - 17x^2 = 31x^2$

EXERCISE 2.2

1. Calculate:
- a) $-73 - 110$
- b) $-27 + 15$
- c) $49 - 62 - 12$
- d) $-55 + 72 - 104$
- e) $-62 - 89 - 32$
- f) $43 - 12 - 105 + 70$

2. Simplify the following:
- a) $-15y + 6y$
- b) $-109a - 21a$
- c) $20ab - 31ab + 62ab$
- d) $11a - 19b + 22a - 12b$
- e) $-45x^2 + 37x + 14x^2 - 52x$
- f) $-15a^2b - 19ab + 44a^2b - 61ab$

3. Write down the value of x that makes each of the following true:
- a) $-510 + 302 = x$
- b) $-150 + x = 0$
- c) $14 - 65 = -25 + x$
- d) $-86 - x = -110$
- e) $7x - 2x = 35$
- f) $5x = -85$

4. Which of the following are like terms?
- a) $3bc; 2ab; -12ac; 8ab$
- b) $x^2y; -3xy; 13x; 12y; 9xy; 4xy^2$
- c) $11ab; 10a^2b; -6ab^2; -a^2b$
- d) $2(3xy^2); -4(5x^2y); -6(4xy); -x(xy)$
- e) $a(ab); b(ab); 2a(ba); -3b(2a)$
- f) $3x(2y^2); 2x(4y); -5y(xy); 12y(3x^2)$

Multiply and divide terms with integer coefficients

In Grade 8 you learned that multiplying a positive integer with a negative integer results in a negative answer and that multiplying two negative integers results in a positive answer.

If you multiply two numbers with the ***same*** sign, the answer will be positive.
If you multiply two numbers with ***different*** signs, the answer will be negative.

Division is the inverse of multiplication so the same rules apply for dividing integers:

If you divide two numbers with the ***same*** sign, the answer will be positive.
If you divide two numbers with ***different*** signs, the answer will be negative.

Study the table of multiplication and division rules for integers:

$(+) \times (+) = +$	$(+) \div (+) = +$	multiplying or dividing two positive numbers gives a positive answer
$(+) \times (-) = -$	$(+) \div (-) = -$	multiplying or dividing a positive and a negative number gives a negative answer
$(-) \times (+) = -$	$(-) \div (+) = -$	multiplying or dividing a negative and a positive number gives a negative answer
$(-) \times (-) = +$	$(-) \div (-) = +$	multiplying or dividing two negative numbers gives a positive answer

Any terms may be multiplied or divided. Use the integer rules to multiply or divide the coefficients and use the exponent rules to multiply or divide the variables.

Example

Simplify:

a) $(-4x)(-3y)$

b) $(-24x^2y) \div (8xy^2)$

Answers

a) $(-4x)(-3y) = 12xy$ ← $(-4)(-3) = +12$ then multiply the variables

b) $(-24x^2y) \div (8xy^2)$

$= -\frac{3x}{y}$ ← $-24 \div 8 = -3$ and $\frac{x^2y}{xy^2} = \frac{x}{y}$

EXERCISE 2.3

1. Calculate the following:
   - a) -144×-20
   - b) $-125 \div -5$
   - c) -200×-30
   - d) $-190 \times -20 \div -100$
   - e) $-360 \div -60 \times 2$
   - f) $-96 \div -12$
2. Simplify the following:
   - a) $-15y \div 3y$
   - b) $-100a^2 \div 20a$
   - c) $2x^2 \times 3x \times -6x$
   - d) $-42xy^2 \div 7xy$
   - e) $-45x^2 \div 3x + 14x^2 \div 7x$
   - f) $35a^2b \times 2ab \div 7ab^2 - 10a^2$
3. Simplify:
   - a) $\frac{32abc}{64ac} \times \frac{8bc}{4b^2}$
   - b) $\frac{16x^2y}{2xy} \times \frac{24xy^2}{-12x}$
   - c) $\frac{12x}{y} \div \frac{6x}{y}$
   - d) $\frac{72mnp}{-9n} \div \frac{18mp}{36m^2}$
4. Find the values of the following:
   - a) $-5 \times 4 + 3 \times -6$
   - b) $12 \times -1 + 20 \times -2$
   - c) $16 \div -8 + 12 \div -3$
   - d) $-10 \times -7 - 22 \div -2$
   - e) $-2 \times 14 - 3 \times -5$
   - f) $84 \div -12 + 56 \div 8$
5. Simplify the following:
   - a) $-6a \times 4b + 2b \times -7a$
   - b) $11m \times -1n + 2m \times -6n$
   - c) $16y^2 \div -8y + 15y \div -5$
   - d) $-12q \times -6p - 18pq \div -3$
   - e) $-3ab \times 4bc - 2ac \times -6b^2$
   - f) $4mn \div -2m + 50n^2 \div 5n$

Mixed operations with integers

When you do a calculation that has mixed operations, the order of operations is sometimes important and can affect the answer. If you perform the calculations in the wrong order, you will get an incorrect answer.

When you add or multiply like terms the order of the numbers does not matter, for example: $-84x - 25x = -109x = -25x - 84x$ and $-12a \times -6a = 72a^2 = -6a \times -12a$

When numbers in brackets are multiplied by a number in front of the bracket, each number in the bracket is affected. This distributive property of numbers works for addition and subtraction, for example:
$-3(5a + 6a) = (-3 \times 5a) + (-3 \times 6a) = -15a + (-18a) = -33a$

If you simplified the brackets first, you would get the same answer:
$-3(5a + 6a) = -3(11a) = -3 \times 11a = -33a$

The correct order of operations is:

- First do calculations inside brackets, where possible.
- Next multiply and divide, working from left to right.
- Finally, add and subtract, working from left to right.
- You may only add and subtract constants or like terms.

EXERCISE 2.4

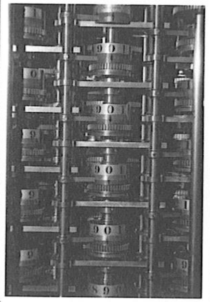
1. Multiply out the brackets and then simplify:
   - a) $-4(5a - 6b)$
   - b) $12(-10x + 5y)$
   - c) $-2(15m) + 3(-12m)$
   - d) $11(-7p) - 6(-8p)$
   - e) $2a(-3a + 12c)$
   - f) $-5b(6a - 5)$
2. Calculate:
   - a) $-18 + 6 \times (-3)$
   - b) $12 \div (-4) + 6$
   - c) $24 \times 3 - 20$
   - d) $100 \times (-2) + 50 \div (-5)$
   - e) $-10 \times 2 - 6 \div -3$
   - f) $-12 + 14 \div (-7) + 28 \div 4$
3. Simplify:
   - a) $12a(4a - 6b) + 9b(8a - 4)$
   - b) $(-2a \times 4b) - (-12a \times 6b)$
   - c) $-20(8x + 12y) - 8(12y - 10x)$
   - d) $-20m \times 8n + 20m \times (-12n)$
   - e) $2x(x - y) - 3y(y - x)$
   - f) $-3a(a - 2b) + 4b(2a - 1)$
4. Simplify:
   - a) $y^2 - (-y)^2$
   - b) $2x(3x) - 3x(-2x)$
   - c) $-a(a + 1) - 2(3 - a) + a(a - 2)$
   - d) $-4m(2n^2) \div (-2n)^2 + 3m$
   - e) $3b(2c - 3d) - 6c(b + d) + 3d(b + c)$
   - f) $3xy(2 - x) - 2y(3x + 1) - y(-3x^2 - 2)$

Did you know?

The average surface temperature in Antarctica is −37 °C. Scientists predict that rising temperatures will cause ice shelves to break up and sea levels to rise.

Did you know?

One of the first mechanical calculating devices was devised by Charles Babbage who worked on it from 1821 to 1832. His "difference engine" was over 2 m tall and weighed about 15 tonnes. This was the first successful automatic calculator.



Squares, cubes, square roots and cube roots

Squares, cubes and powers

When any integer is squared, the answer is always positive, for example $(-12)^2 = (-12)(-12) = +144$ and $12^2 = 12 \times 12 = 144$

Note that two different numbers squared can give the same answer. This will be true for the square of any integer and the square of its additive inverse.

An integer raised to any even power will result in a positive answer:

$5^4 = 5 \times 5 \times 5 \times 5 = 625$

$(-5)^4 = (-5)(-5)(-5)(-5) = 625$

A power with an even exponent is always positive.

Note that $(-12)^2$ and -12^2 give different answers.

$(-12)^2 = 144$ as (-12) is squared $(-12)(-12) = 144$ but -12^2 is equal to $-(12)(12) = -144$.

In the case of -12^2, only the number 12 is squared. The minus sign is not inside a bracket, so is not affected by the power.

The minus sign is only squared when it is inside a bracket.

When any integer is cubed, the result has the same sign as the original number, for example $(-6)^3 = (-6)(-6)(-6) = -216$ but $6^3 = 6 \times 6 \times 6 = 216$

Cube numbers keep the sign of their cube root. The cube number -216 has the same sign as its cube root -6.

Any integer raised to an odd power will have an answer with the same sign as the original number.

A power with a negative base and an odd exponent is always negative.

For example: $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$

Example

Simplify:

a) $-8^2 + (-2)^6$

b) $\frac{(-9)^2}{-3^3}$

Answers

a) $-8^2 + (-2)^6$

$= -64 + 64$ ← use the rules to get the correct signs for each term

$= 0$

b) $\frac{(-9)^2}{-3^3}$

$= \frac{81}{-27}$ ← use the rules to get the correct signs for the numerator and denominator

$= -3$

EXERCISE 2.5

1. Calculate:
   a) $(-2)^3 - 3^2$
   b) $2^5 - (-2)^5$
   c) $4^3 - (-3)^4$
   d) $-1^3 - 2^3 - (-3)^3$
2. Simplify:
   a) $\frac{(-4)^4}{2^6}$
   b) $\frac{7^8}{-7^7}$
   c) $\frac{(-5)^2}{-5^2}$
   d) $\frac{-6^2}{(-3)^2(-2)^2}$
   e) $\frac{(-2)^2(-5)^2}{10^2}$
   f) $\frac{-3^3}{9} + \frac{8^2}{(-4)^2}$
3. Calculate:
   a) $(-11)^2 + (-3)^3 - 9^2$
   b) $3^3 - 2^3 - 5^2 + (-2)^2$
   c) $-8^2 + (-2)^6 + (-6)^2$
   d) $-1^7 \times (-2)^2 + 4^1$
   e) $4^2 \div (-2)^4 + (-1)^{101}$
   f) $(-12)^0 \times 6^2 - 4^3$
4. Calculate $2^{2\,014} - 2^{2\,013} \times 2$

Square roots and cube roots

In algebra, if you solve the equation $a^2 = 144$, there will be two correct answers for a.

For example $(-12)^2 = 144$ and $12^2 = 144$. The numbers are different but the result is the same for the square of both numbers.

The answer to the equation $a^2 = 144$ is written $a = \pm\sqrt{144}$ and the answers are 12 and -12. When there is no $\pm$ sign in front of a square root, there is only one answer and this is the positive square root.

When you solve the equation $\sqrt{144} = a$, there is only one answer: $a = 12$

$\sqrt{x}$ has only one positive answer for $x > 0$

There is no real solution to the square root of a negative integer. This is because there is no number that gives a positive answer when multiplied by itself.

Remember: **(+) × (+) = + and (−) × (−) = +**

You have already seen that any integer cubed results in an answer with the same sign as the base, for example $(-4)^3 = (-4)(-4)(-4) = -64$ but $4^3 = 4 \times 4 \times 4 = 64$

This means that $\sqrt[3]{-64} = -4$ and $\sqrt[3]{64} = 4$

A cube root with a negative base is always negative.

For example: $\sqrt[3]{-27} = \sqrt[3]{(-3)(-3)(-3)} = -3$

Example

Simplify:

a) $-\sqrt[3]{64} - \sqrt[3]{-27}$

b) $\dfrac{\sqrt[3]{-125}\,\sqrt{64}}{-\sqrt{100}\,\sqrt[3]{-64}}$

Answers

a) $-\sqrt[3]{64} - \sqrt[3]{-27}$

$= -4 - (-3)$ ← use the rules to get the correct signs for each term

$= -4 + 3$

$= -1$

b) $\dfrac{\sqrt[3]{-125}\,\sqrt{64}}{-\sqrt{100}\,\sqrt[3]{-64}}$

$= \dfrac{-5(8)}{-10(-4)}$ ← use the rules to get the correct signs for the numerator and denominator

$= \dfrac{-40}{40}$ ← simplify the numerator and denominator

$= -1$

EXERCISE 2.6

1. Simplify:

a) $\sqrt{81}$ b) $-\sqrt{121}$

c) $\sqrt[3]{-1}$ d) $\sqrt[3]{27}$

e) $\sqrt[3]{-64}$ f) $-\sqrt[3]{-125}$

2. Calculate:

a) $(-2)^3 + \sqrt[3]{-64}$ b) $\sqrt[3]{8} - (-1)^8$

c) $\sqrt{25} - (-5)^2 + \sqrt[3]{-27}$ d) $\sqrt[4]{16} + \sqrt[3]{-64} + \sqrt[5]{-32}$

3. True or false?

a) $\sqrt{25} - \sqrt{36} = \sqrt{-6}$ b) $\sqrt{4} \times \sqrt{9} = \sqrt{36}$

c) $\sqrt{-16} = -4$ d) $\sqrt[3]{-27} = -3$

4. Simplify:

a) $\sqrt{3}(\sqrt{3})^3$ b) $\sqrt{2}(\sqrt{2} - \sqrt{8})$

c) $\sqrt{49} - \sqrt[3]{-27} - \sqrt[3]{-64}$ d) $\sqrt{5}(\sqrt{5})^3 - \sqrt{3}(\sqrt{3})^5$

5. a) $\dfrac{\sqrt{81}\sqrt{64}}{\sqrt{16}\sqrt{9}}$ b) $\dfrac{\sqrt[3]{-64}\sqrt{81}}{\sqrt{100}\sqrt[3]{-27}}$

c) $\sqrt{49}\,\sqrt[3]{-27} \div (-3) + 23$ d) $2(\sqrt[3]{-64} + \sqrt{25})$

6. Simplify:

a) $\sqrt{144x^2y^2}$ b) $-\sqrt{\dfrac{x^4}{25}}$

c) $\sqrt[3]{-27y^3}$ d) $\sqrt{\dfrac{25x^2}{16y^2}}$

e) $\sqrt[3]{-\dfrac{64p^3}{8}}$ f) $-\sqrt{\dfrac{49}{m^2}} - \sqrt{\dfrac{81m}{m^3}}$

Challenge

Given the sum:
$(-2)^3 + (-3)^4 + (-4)^5 + (-5)^6$

Do you think the answer will be positive or negative?

Calculate the answer.

Revision

1. Write down the numbers that are integers from the following list of numbers:
$117{,}4$; $-4\ 561\ 234$; $\sqrt[3]{\dfrac{54}{2}}$; $67\ 598$; $\dfrac{5}{8}$; $0{,}125$; $-\dfrac{1}{2}$; $\sqrt{36}$ (1)

2. Write the following numbers in ascending order:
$-9\ 942$; $-16\ 432$; $-33\ 915$; $-20\ 020$; $-8\ 978$ (1)

3. The temperature one night in Ulundi was measured at −1 °C. The temperature increased to 8 °C the next day. By how many degrees did the temperature rise? (1)

4. Which temperature is lower?
−3 °C or −4 °C (1)

5. Calculate the value of the expression $-x^2 + 3x - 2$ when
a) $x = -2$ b) $x = 0$ (2)

6. Calculate:
a) $-64 - 103 + 75$ b) $58 - (-17) + 12$ (2)

7. Simplify the following:
a) $16a^2 - 21b - 32a^2 + 19b$ b) $-5x^2 + 7x - 14x^2 - 32x$ (4)

8. Write down the value of x that makes each of the following true:
a) $-160 + x = 20$ b) $-2x = 48$ (2)

9. Calculate:
a) $-1\ 200 \times (-45)$ b) $-169 \div (-13)$
c) $32 - 14 \times (-3)$ d) $84 \times (-3) - 510 \div (-5)$ (4)

10. Simplify:
a) $(-15x)(-30y^2)$ (2)
b) $(-84x^2y) \div (12xy^2)$ (2)

11. Simplify:
a) $(-3)^3 - \sqrt[3]{-125}$ (2)
b) $\dfrac{-6a^2}{(3a)^2(-2a)^2}$ (2)
c) $-\sqrt{121a^2b^2}$ (2)
d) $\sqrt[3]{\dfrac{-27x^3}{64}}$ (2)

Total: 30 marks