3d Factorising expressions

Factorising is the reverse of expanding.

● To **factorise** an expression you insert brackets.

For example, $8x - 20 = 4(2x - 5)$
Each term is divided by 4.

To fully factorise an expression, the term outside the brackets should be the **highest common factor (HCF)** of all terms inside the brackets.

Example

Factorise
a $6p + 15$ **b** $5x^2 + 2x$ **c** $4mn - 12m^2$

a $6p + 15$
$= 3(2p + 5)$
The HCF of $6p$ and 15 is 3.

b $5x^2 + 2x$
$= x(5x + 2)$
The HCF of $5x^2$ and $2x$ is x.

c $4mn - 12m^2$
$= 4m(n - 3m)$
The HCF of $4mn$ and $12m^2$ is $4m$.

Sometimes, when you expand a pair of double brackets, some of the terms cancel out.
For example,
$(x + 5)(x - 5) = x^2 + 5x - 5x - 25$
$= x^2 - 25$

● $(a + b)(a - b) = a^2 - b^2$
▶ This is often called the **difference of two squares**.

Example

Factorise
a $x^2 - 16$ **b** $4y^2 - 25$ **c** $2p^2 - 18$

a $x^2 - 16$
$= (x + 4)(x - 4)$
You should spot that $\sqrt{16} = 4$.

b $4y^2 - 25$
$= (2y + 5)(2y - 5)$
$\sqrt{4y^2} = 2y$ and $\sqrt{25} = 5$.

c $2p^2 - 18$
$= 2(p^2 - 9)$
$= 2(p + 3)(p - 3)$
Take out the HCF of 2 and then use the difference of two squares.

Exercise 3d

1 Copy and complete these factorisations.
a $2x + 8 = 2(x + \square)$
b $6a - 10 = 2(3a - \square)$
c $12 - 3p = 3(\square - p)$
d $8b + 12 = \square(2b + \square)$
e $5mn + 3m = m(\square + \square)$
f $4a - 7ab = \square(4 - \square)$
g $6x + 18xy = 6x(\square + \square)$
h $15pq - 12p = \square(\square - 4)$

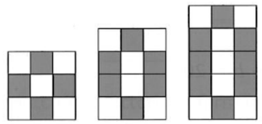
2 Fully factorise these expressions.
a $3x + 12$ **b** $5y - 20$
c $24a - 16$ **d** $21b - 7$
e $14 - 8p$ **f** $10 - 10q$
g $3x + 4xy$ **h** $12mn + 9n$
i $4a + 12b + 20c$
j $25a + 10ab - 15ac$

3 Fully factorise these expressions.
a $x^2 + 2x$ **b** $3y^2 - 4y$
c $5p^2 + p$ **d** $a^2 - ab$
e $2n^2 + 4n$ **f** $12m^2 + 9m$
g $5pq - 15p^2$ **h** $20x^2y - 12x$
i $10a^2 + 20ab + 12a$
j $6x^2 - 12x^2y + 3xy$

4 Use the difference of two squares to factorise these expressions.
a $x^2 - 9$ **b** $x^2 - 25$
c $y^2 - 49$ **d** $y^2 - 100$
e $4x^2 - 25$ **f** $9a^2 - 16$
g $4p^2 - 49$ **h** $16a^2 - 81$
i $x^2 - 4y^2$ **j** $9m^2 - 64$
k $(x + 1)^2 - (x - 1)^2$
l $(x + y)^2 - (x - y)^2$

Problem solving

5 Alexander and William have each written a formula for the number of red squares, r, in a rectangle of height, n.



Alexander wrote the formula $r = 2(n - 1)$
William wrote the formula $r = 2n - 2$
a Use factorisation to explain why they are both right.
b **Justify** each formula by referring to the diagrams.

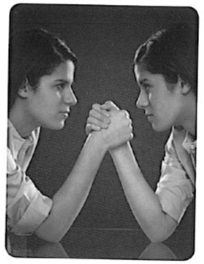
6 Use factorisation to simplify these algebraic fractions.
a $\frac{7x + 14}{21x + 42}$ **b** $\frac{x^2 - y^2}{x + y}$
c $\frac{a^2 - b^2}{2a - 2b}$ **d** $\frac{5m + 5n}{m^2 - n^2}$

7 Fully factorise these expressions.
a $x^4 - 16$ **b** $x^4 - a^4$

Did you know?

In the 9th century, al-Khwarizmi wrote one of the first books on equations. The word algebra comes from the arabic *al-jabr* which means restoration.

In Maths, **identical** has a very specific meaning, more than just "looking the same".



A **formula** shows the connection between two or more **variables**.

Temperatures in Fahrenheit and Celsius are connected by the formula

$F = \frac{9}{5}C + 32$ The variables are

F = temperature in °F

and C = temperature in °C.

An **equation** is true for particular values of the **unknown**.

$9(x - 5) = 27$	is true only when $x = 8$	$9(8 - 5) = 27$
$4(5t - 9) = 3(t + 5)$	is true only when $t = 3$	$4(5 \times 3 - 9) = 3(3 + 5)$

An **identity** is true for all values of the unknown.

▶ $\equiv$ means 'is identically equal to'.

$9(x - 5) \equiv 9x - 45$	is true for all values of x
$4(5t - 9) \equiv 20t - 36$	is true for all values of t

Example

Prove that these are identities.

a $7(a - 9) - 3(a - 4) \equiv 4a - 51$ **b** $(b - 4)(b + 5) \equiv b^2 + b - 20$

a $7(a - 9) - 3(a - 4)$

$= 7a - 63 - 3a + 12$

$= 4a - 51$

$7(a - 9) - 3(a - 4) \equiv 4a - 51$

b $(b - 4)(b + 5)$

$= b^2 - 4b + 5b - 20$

$= b^2 + b + 20$

$(b - 4)(b + 5) \equiv b^2 + b - 20$

Recognising the difference of two squares can sometimes help you with identities.

Example

Prove that these are identities.

a $(x + 5)(x - 5) \equiv x^2 - 25$ **b** $(p - q)(p + q) \equiv p^2 - q^2$

a $(x + 5)(x - 5)$

$= x^2 + 5x - 5x - 25$

$= x^2 - 25$

$(x + 5)(x - 5) \equiv x^2 - 25$

b $(p - q)(p + q)$

$= p^2 - pq + pq - q^2$

$= p^2 - q^2$

$(p - q)(p + q) \equiv p^2 - q^2$

Exercise 3e

1 Find the value of the required variable in each of these formulae.

a $A = l^2$
Find A when $l = 2.5$

b $P = 2l + 2w$
Find P when $l = 4\frac{1}{2}$ and $w = 2\frac{1}{4}$

c $V = \pi r^2 h$
Find V when $r = 3$ and $h = 8$

d $s = ut + \frac{1}{2}at^2$
Find s when $u = 0$, $a = 5.8$ and $t = 2$

2 Solve these equations to find the value of the unknowns.

a $3x + 5 = 23$ **b** $5(y + 4) = 35$

c $25 = 7a - 3$ **d** $20 - 4b = 8$

e $6(5 - p) = 18$

f $9q - 5 = 7q + 5$

g $3(m + 7) = 5m - 3$

h $22 - n = 5(n - 4)$

3 Prove that these are identities.

a $6(a + 5) \equiv 6a + 30$

b $9(b - 6) \equiv 9b - 54$

c $10k + 5(k - 7) \equiv 15k - 35$

d $3(t + 6) + 7(t + 1) \equiv 5(2t + 5)$

e $3(p - 4) + 6(p + 3) \equiv 3(3p + 2)$

f $8(q + 3) - 6(q - 3) \equiv 2(q + 21)$

g $4(2x + 3) - 3(x - 6) \equiv 5(x + 6)$

h $7(2 - y) + 8(5 + 2y) \equiv 9(y + 6)$

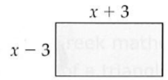
i $(x - 3)(x + 3) \equiv x^2 - 9$

j $(8 + y)(8 - y) \equiv 64 - y^2$

Problem solving

4 Write an algebraic expression for each area. Expand and simplify your answers using the difference of two squares.

a



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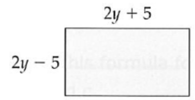

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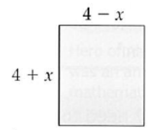


5 Match the factorised form of the difference of two squares to their expanded equivalents.

a $(x + 5)(x - 5)$ **b** $x^2 - 16$ **c** $(3x + 4)(3x - 4)$

d $9x^2 - 16$ **e** $(2x + 5)(2x - 5)$ **f** $4x^2 - 25$

g $x^2 - 25$ **h** $(x + 4)(x - 4)$

6 Write an identity for each of these products.

a $(a + b)^2$ **b** $(a + b)(c + d)$ **c** $(a + b)(c - d)$

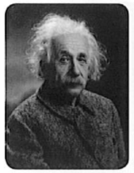
7 Use the difference of two squares to do these calculations.

a $101^2 - 99^2$ **b** $1002^2 - 998^2$ **c** $10003^2 - 9997^2$

d $0.8^2 - 0.2^2$ **e** $8.7^2 - 1.3^2$ **f** $0.1^2 - 0.01^2$

3f Formulae

One of the most famous formulae is Einstein's $E = mc^2$.



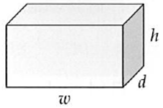
- A **formula** is a relationship or rule expressed in symbols.

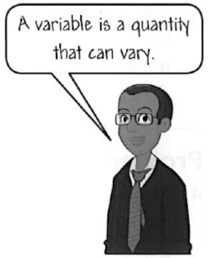
To find the surface area, S, of a cuboid with width w, depth d and height h you can use the formula

$S = 2dw + 2dh + 2hw$.

S, w, d and h are the **variables** in this formula.

You can **substitute** values into this formula to find an unknown variable.

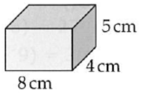




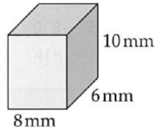
Example

Find the surface area of these cuboids.

a



b



a $S = 2dw + 2dh + 2hw$

$= 2 \times 4 \times 8 + 2 \times 4 \times 5 + 2 \times 5 \times 8$

$= 64 + 40 + 80$

$S = 184\text{ cm}^2$

b $S = 2dw + 2dh + 2hw$

$= 96 + 120 + 160$

$S = 376\text{ mm}^2$

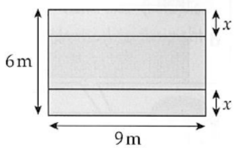
- You may need to **derive** a formula in order to solve a problem.

Example

A garden has a lawn, bordered on either side by flowerbeds.

a Derive a formula for the area, A, of the lawn.

b Use this formula to work out the area of the lawn if each flowerbed is 75 cm wide.



a The width of the lawn is 6 m minus two lots of x m (one for each flowerbed).

$A = lw$

$= 9(6 - 2x)$

$A = 54 - 18x$

b $A = 54 - 18x$

$= 54 - 18 \times 0.75$

$= 40.5\text{ m}^2$

75 cm = 0.75 m

Exercise 3f

1 If $I = \frac{V}{f}$ calculate I when

a $V = 10$ and $f = 2$

b $V = 5$ and $f = 0.5$

c $V = 14$ and $f = -4$

2 If $s = ut + \frac{1}{2}at^2$ calculate s when

a $u = 5$, $t = 2$ and $a = 10$

b $u = -2$, $t = 3$ and $a = 4$

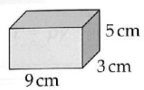
c $u = 1$, $t = -6$ and $a = 2$

Take care with the order of operations!

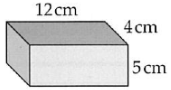
Problem solving

3 Use the formula $S = 2dw + 2dh + 2hw$ to find the surface areas of these cuboids.

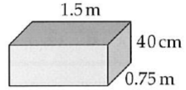
a



b



c



4 The area of an ellipse is given by the formula

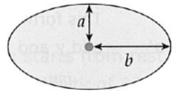
$A = \pi ab$.

Use this formula to find the area of an ellipse when

a $a = 6$ cm and $b = 4$ cm

b $a = 8.2$ cm and $b = 3.7$ cm.

Give your answers in terms of π.



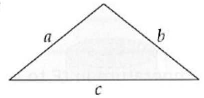
Did you know?

Hero of Alexandria was an ancient Greek mathematician and inventor. Among his many inventions are the first steam engine and the first vending machine.

5 Hero, a Greek mathematician, derived this formula for the area of a triangle with sides a, b and c

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a + b + c)$.



Use Hero's formula to find the area of a triangle with sides

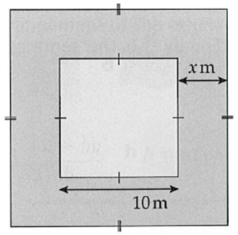
a 3 cm, 4 cm and 5 cm

b 4 cm, 4 cm and 6 cm.

6 A garden consists of a square lawn of length 10 m bordered by a path of width x m.

a Explain why the total length of the garden is $(10 + 2x)$ m.

b Show that the area, A, of the garden is given by the formula $A = 4x^2 + 40x + 100$.



3g Changing the subject of a formula 1

The word 'subject' has different meanings in the English language. You can be the subject of an essay, the subject of a king or the subject of an equation!

You **change the subject** of a formula by rearranging it.

In the formula $y = mx + c$ — y is the subject.

An equivalent formula is $x = \frac{y - c}{m}$ — x is the subject.

You can use **inverse operations** to rearrange a formula.

Example

Rearrange these formulae to make x the subject.

a $a = p + qx$ **b** $m = \frac{x + y}{n}$

a $a = p + qx$

This formula says 'start with x, multiply by q and add p to get a'.

$a - p = qx$ Subtract p.

$\frac{a - p}{q} = x$ Divide by q.

$x = \frac{a - p}{q}$

b $m = \frac{x + y}{n}$

This formula says 'start with x, add y and divide by n to get m'.

$mn = x + y$ Multiply by n.

$mn - y = x$ Subtract y.

$x = mn - y$

Example

A formula that connects temperature in °F to temperature in °C is $F = \frac{9}{5}C + 32$. Make C the subject of this formula and then convert 64.4 °F to °C.

$F = \frac{9}{5}C + 32$

$F - 32 = \frac{9}{5}C$ Subtract 32

$\frac{5}{9}(F - 32) = C$ Dividing by $\frac{9}{5}$ is the same as multiplying by $\frac{5}{9}$

$C = \frac{5}{9}(F - 32)$

$C = \frac{5}{9}(64.4 - 32) = 18$

64.4°F is 18°C

Exercise 3g

1 Make x the subject of these formulae.

a $x + p = q$ **b** $5y = x + y$
c $x + a = b + c$ **d** $k = x - t$
e $x - g^2 = h$ **f** $p = x - mn$
g $x + \sqrt{a} = b^2$ **h** $k + x = k + t^2$

2 Make y the subject of these formulae.

a $ay = b$ **b** $\frac{y}{p} = x$
c $xy = w + z$ **d** $\frac{y}{a} = b + c$
e $r = py + q$ **f** $ky - t^2 = x$
g $\frac{y}{g + h} = k$ **h** $\frac{y}{m} - x = n$
i $xyz = k^2$ **j** $aby = m + n$
k $p + mny = q - p$
l $y(a - b) = c^2$

3 Make a the subject of these formulae.

a $x = y(a + b)$ **b** $q(a - r) = p$
c $a(m + n) = k$ **d** $t^2(a - v) = w$
e $S = 2\pi(a + b)$ **f** $y = \frac{1}{2}(a - x)$
g $m = \frac{1}{10}(mn + a)$ **h** $k^2 = \frac{1}{\pi}(a - h)$

Problem solving

4 The formula $v = u + at$ connects the variables

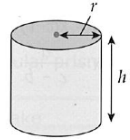
v = final velocity a = acceleration
u = initial velocity t = time.

Find the acceleration, a, of a train which starts from rest and passes a station 22 seconds later with a velocity of 33 m/s. (If an object starts from rest then its initial velocity is zero.)

5 The surface area of a closed cylinder is given by the formula

$S = 2\pi r(h + r)$

where r = radius and h = height.



a Show that $h = \frac{S - 2\pi r^2}{2\pi r}$.

b Find the height of a cylinder with a surface area of 962π cm² and a radius of 13 cm. Leave π in your working.

6 Match these formulae so that one is a rearrangement of the other.

a $a = b(x + y)$ **b** $x = \frac{a - y}{b}$ **c** $x = \frac{b + ay}{a}$ **d** $y = a - bx$

e $b = ax + y$ **f** $x = \frac{b - y}{a}$ **g** $x = \frac{a - by}{b}$ **h** $b = a(x - y)$

3h Changing the subject of a formula 2

If you can change the subject of these formulae, you'll be on your way to being really good at algebra!

Example

Make x the subject of these formulae.

a $b = a - x$ **b** $p = \frac{q}{x}$ **c** $x^2 = t$

a
$b = a - x$
$b + x = a$
$x = a - b$

b
$p = \frac{q}{x}$
$px = q$
$x = \frac{q}{p}$

c
$x^2 = t$
$x = \pm\sqrt{t}$ Square root both sides.

Here are some guidance tips that might help:

- If the intended subject is negative, add terms to make it positive.
- If it appears in a **denominator**, multiply to undo the fraction.
- If it appears squared, you will need to take the square root.

Example

Make x the subject of these formulae.

a $m = p - qx$ **b** $\frac{a}{x} + b = c$ **c** $x^2 - k = t$

a
$m = p - qx$
$m + qx = p$
$qx = p - m$
$x = \frac{p - m}{q}$

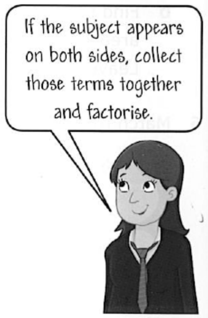
b
$\frac{a}{x} + b = c$
$\frac{a}{x} = c - b$
$a = x(c - b)$
$x = \frac{a}{c - b}$

c
$x^2 - k = t$
$x^2 = k + t$
$x = \pm\sqrt{k + t}$

Example

Make x the subject of the formula $ax + b = px + q$

$ax - px = q - b$ Collect the terms in x on one side of the formula.
$x(a - p) = q - b$ Factorise to isolate the term in x.
$x = \frac{q - b}{a - p}$ Divide both sides by $a - p$.



Exercise 3h

1 Make x the subject of these formulae.
- **a** $q = p - x$ **b** $g = h - x$
- **c** $t = mn - x$ **d** $r^2 = c - x$
- **e** $a = b - xy$ **f** $t = v - wx$
- **g** $p = k(y - x)$ **h** $r = \frac{1}{\pi}(a - x)$

2 Make y the subject of these formulae.
- **a** $\frac{a}{y} = c$ **b** $t = \frac{k}{y}$
- **c** $\frac{p}{y} = \frac{q}{r}$ **d** $e = \frac{x}{yz}$
- **e** $\frac{t}{y} + r = s$ **f** $a = \frac{b}{y} - c$
- **g** $k^2 = t + \frac{x}{y}$ **h** $\frac{c}{y} - n = -m$
- **i** $\frac{f}{g + y} = h$ **j** $x = \frac{a}{y - b}$
- **k** $p = q - \frac{r}{y}$ **l** $n = x - \frac{m}{y}$

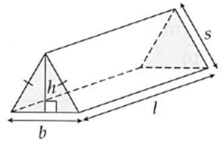
3 Make p the subject of these formulae.
- **a** $p^2 = t$
- **b** $p^2 - q = k$
- **c** $m = p^2 - n^2$
- **d** $\sqrt{p} = x$
- **e** $ap^2 = b$
- **f** $p^2x = y$
- **g** $t - p^2 = k$
- **h** $f = g - p^2$

4 Rearrange these formula with x on both sides to make x the subject.
- **a** $ax + b = cx + d$
- **b** $px + q = r - tx$
- **c** $hx - g = m - nx$
- **d** $p - qx = r - sx$

Problem solving

5 The surface area of an isosceles triangular prism is given by this formula.

$A = bh + 2ls + lb$.



a Find the surface area, A, of an isosceles triangular prism with $b = 6$ cm, $h = 4$ cm, $l = 10$ cm and $s = 5$ cm.

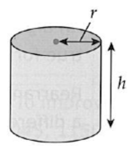
b Rearrange the formula $A = bh + 2ls + lb$ to make

i s the subject **ii** b the subject.

6 The diagram shows a cylinder with radius r and height h.

a Try to find a formula for the **volume** of the cylinder.

b Rearrange your formula to make r the subject.



7 The formula for the volume of a sphere is

$V = \frac{4}{3}\pi r^3$

Rearrange the formula to make r the subject.